On cluster C^*-algebras

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Abstract

We introduce a C^*-algebra $\mathbb{A}(\mathbf{x}, Q)$ attached to the cluster $\mathbf{x}$ and a quiver Q. If Q_T is the quiver coming from a triangulation T of the Riemann surface S with a finite number of cusps, we prove that the primitive spectrum of $\mathbb{A}(\mathbf{x}, Q_T)$ times $\mathbb{R}$ is homeomorphic to a generic subset of the Teichmüller space of surface S. We conclude with an analog of the Tomita-Takesaki theory and the Connes invariant $T(\mathcal{M})$ for the algebra $\mathbb{A}(\mathbf{x}, Q_T)$.

Key words and phrases: cluster algebras, Riemann surfaces, C^*-algebras

MSC: 13F60 (cluster algebras); 14H55 (Riemann surfaces); 46L85 (noncommutative topology)

1 Introduction

Cluster algebras of rank m are a class of commutative rings introduced by [Fomin & Zelevinsky 2002] [10]. Among these algebras one finds coordinate rings of important algebraic varieties, like the Grassmannians and Schubert varieties; cluster algebras appear in the Teichmüller theory [Fomin, Shapiro & Thurston 2008] [9]. Unlike the coordinate rings, the set of generators x_i of cluster algebra is usually infinite and defined by induction from a *cluster* $\mathbf{x} = (x_1, \dots, x_m)$ and a *quiver* Q, see [Williams 2014] [22] for an excellent survey; the cluster algebra is denoted by $\mathcal{A}(\mathbf{x}, Q)$. Notice that the $\mathcal{A}(\mathbf{x}, Q)$ has an additive structure of countable (unperforated) abelian group with an order satisfying the Riesz interpolation property; see Remark 3. In other words, the cluster algebra $\mathcal{A}(\mathbf{x}, Q)$ is a *dimension group* by the Effros-Handelman-Shen Theorem [Effros 1981, Theorem 3.1] [7].

The subject of our note is an operator algebra $\mathbb{A}(\mathbf{x},Q)$, such that $K_0(\mathbb{A}(\mathbf{x},Q)) \cong \mathcal{A}(\mathbf{x},Q)$; here $K_0(\mathbb{A}(\mathbf{x},Q))$ is the dimension group of $\mathbb{A}(\mathbf{x},Q)$ and $\cong$ is an isomorphism of the ordered abelian groups [Blackadar 1986, Chapter 7] [2]. The $\mathbb{A}(\mathbf{x},Q)$ is an Approximately Finite C^*-*algebra* (AF-algebra) given by a Bratteli diagram derived explicitly from the pair $(\mathbf{x},Q)$. The AF-algebras were introduced and studied by [Bratteli 1972] [4]; we refer to $\mathbb{A}(\mathbf{x},Q)$ as a *cluster* C^*-*algebra*.

An exact definition of $\mathbb{A}(\mathbf{x},Q)$ can be found in Section 2.4; to give an idea, recall that the pair $(\mathbf{x},Q)$ is called a *seed* and the cluster algebra $\mathcal{A}(\mathbf{x},Q)$ is generated by seeds obtained via *mutation* of $(\mathbf{x},Q)$ (and its mutants) in all *directions* k, where $1 \le k \le m$ [Williams 2014, p.5] [22]. The mutation process can be described by an oriented regular tree $\overrightarrow{\mathbb{T}}_m$; the vertices of $\overrightarrow{\mathbb{T}}_m$ correspond to the seeds and the outgoing edges to the mutations in directions k. The quotient $\mathfrak{B}(\mathbf{x},Q)$ of $\overrightarrow{\mathbb{T}}_m$ by a relation identifying equivalent seeds at the same level of $\overrightarrow{\mathbb{T}}_m$ is a graph with cycles. (For a quick example of such a graph, see Figure 3.) The cluster C^*-algebra $\mathbb{A}(\mathbf{x},Q)$ is an AF-algebra given by the $\mathfrak{B}(\mathbf{x},Q)$ regarded as a Bratteli diagram [Bratteli 1972] [4].

Let $S_{g,n}$ be a Riemann surface of genus $g \ge 0$ with $n \ge 1$ cusps and such that $2g-2+n>0$; denote by $T_{g,n} \cong \mathbb{R}^{6g-6+2n}$ the (decorated) Teichmüller space of $S_{g,n}$, i.e. a collection of all Riemann surfaces of genus g with n cusps endowed with the natural topology [Penner 1987] [19]. In what follows, we focus on the algebras $\mathbb{A}(\mathbf{x},Q_{g,n})$ with quivers $Q_{g,n}$ coming from an ideal triangulation of $S_{g,n}$; the corresponding cluster algebra $\mathcal{A}(\mathbf{x},Q_{g,n})$ of rank $m = 6g-6+3n$ is related to the *Penner coordinates* in $T_{g,n}$ [Fomin, Shapiro & Thurston 2008] [9].

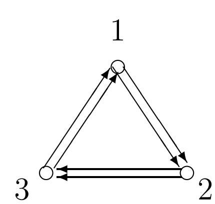

Figure 1: The Markov quiver $Q_{1,1}$.

Example 1 Let $S_{1,1}$ be a once-punctured torus. The ideal triangulation of

$S_{1,1}$ defines the Markov quiver [1] $Q_{1,1}$ shown in Figure 1, see [Fomin, Shapiro & Thurston 2008, Example 4.6] [9]. The corresponding cluster C^*-algebra $\mathbb{A}(\mathbf{x}, Q_{1,1})$ of rank 3 can be written as:

$$\mathbb{A}(\mathbf{x}, Q_{1,1}) \cong \mathfrak{M}/I_0, \tag{1}$$

where I_0 is a primitive ideal of an AF-algebra $\mathfrak{M}$. The unital AF-algebra $\mathfrak{M}$ was originally defined by [Mundici 1988, Section 3] [11]; the genuine notation for such an algebra was $\mathfrak{M}_1$, because $K_0(\mathfrak{M}_1) = (M_1, 1) :=$ free one-generator unital ℓ-group, i.e. a finitely piecewise affine linear continuous real-valued functions on $[0, 1]$ with integer coefficients. The $\mathfrak{M}_1$ was subsequently rediscovered after two decades by [Boca 2008] [3] and denoted by $\mathfrak{A}$. The remarkable properties of $\mathfrak{M}_1$ include the following features. Every primitive ideal of $\mathfrak{M}_1$ is essential [Mundici 2011, Theorem 4.2] [13]. The $\mathfrak{M}_1$ is equipped with a faithful invariant tracial state [Mundici 2009, Theorem 3.1] [12]. The center of $\mathfrak{M}_1$ coincides with the C^*-algebra $C[0, 1]$ of continuous complex valued functions on $[0, 1]$ [Boca 2008, p. 976] [3]. There is an affine weak $*$-homeomorphism of the state space of $C[0, 1]$ onto the space of tracial states on $\mathfrak{M}_1$ [Mundici 2011, Theorem 4.5] [13]. Any state of $C[0, 1]$ has precisely one tracial extension to $\mathfrak{M}_1$ [Eckhardt 2011, Theorem 2.5] [6]. The automorphism group of $\mathfrak{M}_1$ has precisely two connected components [Mundici 2011, Theorem 4.3] [13]. The Gauss map – a Bernoulli shift for continued fractions – is generalized in [Eckhardt 2011] [6] to the noncommutative framework of $\mathfrak{M}_1$. In the light of the original definition of $\mathfrak{M}_1$ and the fact that the K_0-functor preserves exact sequences (see, e.g. [Effros 1981, Theorem 3.1] [7]), the primitive spectrum of $\mathfrak{M}_1$ and its hull-kernel topology is widely known to the lattice-ordered group theorists and the MV-algebraists long ago before the laborious analysis in [Boca 2008] [3], where $\mathfrak{M}_1$ is defined in terms of the Bratteli diagram. We refer the reader to the final part of a paper by [Panti 1999] [18] for a general result encompassing the characterization of the prime spectrum of $(M_1, 1) \cong Prim\ \mathfrak{M}_1$. Moreover, the AF-algebras A_θ introduced by [Effros & Shen 1980] [8] are precisely the infinite-dimensional simple quotients of $\mathfrak{M}_1$; this fact was first proved by [Mundici 1988, Theorem 3.1(i)] [11] and rediscovered independently by [Boca 2008] [3]. Summing up the above, the primitive ideals $I_\theta \subset \mathfrak{M}$ are indexed by numbers $\theta \in \mathbb{R}$; if θ is irrational, the quotient $\mathfrak{M}/I_\theta \cong A_\theta$, where A_θ is the Effros-Shen algebra. In view of

[1] Such a quiver is related to solutions in the integer numbers of the equation $x_1^2 + x_2^2 + x_3^2 = 3x_1x_2x_3$ considered by A. A. Markov; hence the name.

(1), the algebra $\mathfrak{M}$ is a non-commutative coordinate ring of the Teichmüller space $T_{1,1}$. Moreover, there exists an analog of the Tomita-Takesaki theory of modular automorphisms $\{\sigma_t \mid t \in \mathbb{R}\}$ for algebra $\mathfrak{M}$, see Section 4; such automorphisms correspond to the Teichmüller geodesic flow on $T_{1,1}$ [Veech 1986] [21]. The $\sigma_t(I_\theta)$ is an ideal of $\mathfrak{M}$ for all $t \in \mathbb{R}$, where $\sigma_0(I_\theta) = I_\theta$. The quotient algebra $\mathfrak{M}/\sigma_t(I_\theta)$ can be viewed as a non-commutative coordinate ring of the Riemann surface $S_{1,1}$; in particular, the pairs (θ, t) are coordinates in the space $T_{1,1} \cong \mathbb{R}^2$. We refer the reader to [14] for a construction of the corresponding functor.

Motivated by Example 1, denote by $\mathbb{A}(\mathbf{x}, Q_{g,n})$ the cluster C^*-algebra corresponding to a quiver $Q_{g,n}$; let $\sigma_t : \mathbb{A}(\mathbf{x}, Q_{g,n}) \to \mathbb{A}(\mathbf{x}, Q_{g,n})$ be the Tomita-Takesaki flow on $\mathbb{A}(\mathbf{x}, Q_{g,n})$, see Section 4 for the details. Denote by $Prim\ \mathbb{A}(\mathbf{x}, Q_{g,n})$ the set of all primitive ideals of $\mathbb{A}(\mathbf{x}, Q_{g,n})$ endowed with the Jacobson topology and let $I_\theta \in Prim\ \mathbb{A}(\mathbf{x}, Q_{g,n})$ for a generic value of index $\theta \in \mathbb{R}^{6g-7+2n}$. Our main result can be stated as follows.

Theorem 1 *There exists a homeomorphism*

$$h : Prim\ \mathbb{A}(\mathbf{x}, Q_{g,n}) \times \mathbb{R} \to \{U \subseteq T_{g,n} \mid U \text{ is generic}\} \tag{2}$$

given by the formula $\sigma_t(I_\theta) \mapsto S_{g,n}$*; the set* $U = T_{g,n}$ *if and only if* $g = n = 1$*. The* $\sigma_t(I_\theta)$ *is an ideal of* $\mathbb{A}(\mathbf{x}, Q_{g,n})$ *for all* $t \in \mathbb{R}$ *and the quotient algebra* $\mathbb{A}(\mathbf{x}, Q_{g,n})/\sigma_t(I_\theta)$ *is a non-commutative coordinate ring of the Riemann surface* $S_{g,n}$*.*

Remark 1 Theorem 1 is valid for $n \geq 1$, i.e. the Riemann surfaces with at least one cusp. This cannot be improved, since the cluster structure of algebra $\mathbb{A}(\mathbf{x}, Q_{g,n})$ comes from the Ptolemy relations satisfied by the Penner coordinates; so far such coordinates are available only for the Riemann surfaces with cusps [Penner 1987] [19]. It is likely, that the case $n = 0$ has also a cluster structure; we refer the reader to [15], where a functor from the Riemann surfaces $S_{g,0}$ to the AF-algebras $\mathbb{A}(\mathbf{x}, Q_{g,0})/\sigma_t(I_\theta)$ was constructed.

Remark 2 The braid group B_{2g+n} with $n \in \{1, 2\}$ admits a faithful representation by projections in the algebra $\mathbb{A}(\mathbf{x}, Q_{g,n})$; such a construction is based on the *Birman-Hilden Theorem* for the braid groups. This observation and the well-known Laurent phenomenon in the cluster algebra $K_0(\mathbb{A}(\mathbf{x}, Q_{g,n}))$ allow to generalize the Jones and HOMFLY invariants of knots and links to an arbitrary number of variables, see [17] for the details.

The article is organized as follows. We introduce preliminary facts and notation in Section 2. Theorem 1 is proved in Section 3. An analog of the Tomita-Takesaki theory of modular automorphisms and the Connes invariant $T(\mathbb{A}(\mathbf{x}, Q_{g,n}))$ of the cluster C^*-algebra $\mathbb{A}(\mathbf{x}, Q_{g,n})$ is constructed.

2 Notation

In this section we introduce notation and briefly review some preliminary facts. The reader is encouraged to consult [Bratteli 1972] [4], [Fomin, Shapiro & Thurston 2008], [Fomin & Zelevinsky 2002] [10], [Penner 1987] [19] and [Williams 2014] [22] for the details.

2.1 Cluster algebras

A *cluster algebra* $\mathcal{A}$ of rank m is a subring of the field $\mathbb{Q}(x_1, \dots, x_m)$ of rational functions in n variables. Such an algebra is defined by a pair $(\mathbf{x}, B)$, where $\mathbf{x} = (x_1, \dots, x_m)$ is a cluster of variables and $B = (b_{ij})$ is a skew-symmetric integer matrix; the new cluster $\mathbf{x}'$ is obtained from $\mathbf{x}$ by an excision of the variable x_k and replacing it by a new variable x'_k subject to an exchange relation:

$$x_k x'_k = \prod_{i=1}^{m} x_i^{\max(b_{ik},0)} + \prod_{i=1}^{m} x_i^{\max(-b_{ik},0)}. \tag{3}$$

Since the entries of matrix B are exponents of the monimials in cluster variables, one gets a new pair $(\mathbf{x}', B')$, where $B' = (b'_{ij})$ is a skew-symmetric with:

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise.} \end{cases} \tag{4}$$

For brevity, the pair $(\mathbf{x}, B)$ is called a *seed* and the seed $(\mathbf{x}', B') := (\mathbf{x}', \mu_k(B))$ is obtained from $(\mathbf{x}, B)$ by a *mutation* μ_k in the direction k, where $1 \le k \le m$; the μ_k is an involution, i.e. $\mu_k^2 = Id$. The matrix B is called *mutation finite* if only finitely many new matrices can be produced from B by repeated matrix mutations. The cluster algebra $\mathcal{A}(\mathbf{x}, B)$ can be defined as the subring of $\mathbb{Q}(x_1, \dots, x_m)$ generated by the union of all cluster variables obtained from the initial seed $(\mathbf{x}, B)$ by mutations of $(\mathbf{x}, B)$ (and its iterations) in all possible directions. We shall write $\overrightarrow{\mathbb{T}}_m$ to denote an oriented tree whose vertices are seeds $(\mathbf{x}', B')$ and m outgoing arrows in each vertex correspond to mutations μ_k of the seed $(\mathbf{x}', B')$. The *Laurent phenomenon* proved by [Fomin &

Zelevinsky 2002] [10] says that $\mathcal{A}(\mathbf{x}, B) \subset \mathbb{Z}[\mathbf{x}^{\pm 1}]$, where $\mathbb{Z}[\mathbf{x}^{\pm 1}]$ is the ring of the Laurent polynomials in variables $\mathbf{x} = (x_1, \ldots, x_n)$; in other words, each generator x_i of algebra $\mathcal{A}(\mathbf{x}, B)$ can be written as a Laurent polynomial in n variables with the integer coefficients.

Remark 3 The Laurent phenomenon turns the additive structure of cluster algebra $\mathcal{A}(\mathbf{x}, B)$ into a totally ordered abelian group satisfying the Riesz interpolation property, i.e. a dimension group [Effros 1981, Theorem 3.1] [7]; the abelian group with order comes from the semigroup of the Laurent polynomials with *positive* coefficients, see [16] for the details. A background on the partially and totally ordered, unperforated abelian groups with the Riesz interpolation property can be found in [Effros 1981] [7].

To deal with mutation formulas (3) and (4) in geometric terms, recall that a *quiver* Q is an oriented graph given by the set of vertices Q_0 and the set of arrows Q_1; an example of quiver is given in Figure 1. Let k be a vertex of Q; the mutated at vertex k quiver $\mu_k(Q)$ has the same set of vertices as Q but the set of arrows is obtained by the following procedure: (i) for each sub-quiver $i \to k \to j$ one adds a new arrow $i \to j$; (ii) one reverses all arrows with source or target k; (iii) one removes the arrows in a maximal set of pairwise disjoint 2-cycles. The reader can verify, that if one encodes a quiver Q with n vertices by a skew-symmetric matrix $B(Q) = (b_{ij})$ with b_{ij} equal to to the number of arrows from vertex i to vertex j, then mutation μ_k of seed $(\mathbf{x}, B)$ coincides with such of the corresponding quiver Q. Thus the cluster algebra $\mathcal{A}(\mathbf{x}, B)$ is defined by a quiver Q; we shall denote such an algebra by $\mathcal{A}(\mathbf{x}, Q)$.

2.2 Cluster algebras from Riemann surfaces

Let g and n be integers, such that $g \geq 0$, $n \geq 1$ and $2g - 2 + n > 0$. Denote by $S_{g,n}$ a Riemann surface of genus g with the n cusp points. It is known that the fundamental domain of $S_{g,n}$ can be triangulated by $6g - 6 + 3n$ geodesic arcs γ, such that the footpoints of each arc at the absolute of Lobachevsky plane $\mathbb{H} = \{x + iy \in \mathbb{C} \mid y > 0\}$ coincide with a (pre-image of) cusp of the $S_{g,n}$. If $l(\gamma)$ is the hyperbolic length of γ measured (with a sign) between two horocycles around the footpoints of γ, then we set $\lambda(\gamma) = e^{\frac{1}{2} l(\gamma)}$; the $\lambda(\gamma)$ are known to satisfy the *Ptolemy relation*:

$$\lambda(\gamma_1)\lambda(\gamma_2) + \lambda(\gamma_3)\lambda(\gamma_4) = \lambda(\gamma_5)\lambda(\gamma_6), \tag{5}$$

where $\gamma_1, \dots, \gamma_4$ are pairwise opposite sides and γ_5, γ_6 are the diagonals of a geodesic quadrilateral in $\mathbb{H}$.

Denote by $T_{g,n}$ the decorated Teichmüller space of $S_{g,n}$, i.e. the set of all complex surfaces of genus g with n cusps endowed with the natural topology; it is known that $T_{g,n} \cong \mathbb{R}^{6g-6+2n}$.

Theorem 2 ([Penner 1987] [19]) *The map λ on the set of $6g-6+3n$ geodesic arcs γ_i defining a triangulation of $S_{g,n}$ is a homeomorphism with the image $T_{g,n}$.*

Remark 4 Notice that among $6g-6+3n$ real numbers $\lambda(\gamma_i)$ there are only $6g-6+2n$ independent, since such numbers must satisfy n Ptolemy relations (5).

Let T be a triangulation of surface $S_{g,n}$ by $6g-6+3n$ geodesic arcs γ_i; consider a skew-symmetric matrix $B_T = (b_{ij})$, where b_{ij} is equal to the number of triangles in T with sides γ_i and γ_j in clockwise order minus the number of triangles in T with sides γ_i and γ_j in the counter-clockwise order. It is known that matrix B_T is always mutation finite. The cluster algebra $\mathcal{A}(\mathbf{x}, B_T)$ of rank $6g-6+3n$ is called *associated* to triangulation T.

Example 2 Let $S_{1,1}$ be a once-puncuted torus of Example 1. The triangulation T of the fundamental domain $(\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2$ of $S_{1,1}$ is sketched in Figure 2 in the charts $\mathbb{R}^2$ and $\mathbb{H}$, respectively. It is easy to see that in this case $\mathbf{x} = (x_1, x_2, x_3)$ with $x_1 = \gamma_{23}, x_2 = \gamma_{34}$ and $x_3 = \gamma_{24}$, where γ_{ij} denotes a geodesic arc with the footpoints i and j. The Ptolemy relation (5) reduces to $\lambda^2(\gamma_{23}) + \lambda^2(\gamma_{34}) = \lambda^2(\gamma_{24})$; thus $T_{1,1} \cong \mathbb{R}^2$. The reader is encouraged to verify, that matrix B_T has the form:

$$B_T = \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}. \tag{6}$$

Theorem 3 ([Fomin, Shapiro & Thurston 2008] [9]) *The cluster algebra $\mathcal{A}(\mathbf{x}, B_T)$ does not depend on triangulation T, but only on the surface $S_{g,n}$; namely, replacement of the geodesic arc γ_k by a new geodesic arc γ_k' (a flip of γ_k) corresponds to a mutation μ_k of the seed $(\mathbf{x}, B_T)$.*

Remark 5 In view of Theorems 2 and 3, the $\mathcal{A}(\mathbf{x}, B_T)$ corresponds to an algebra of functions on the Teichmüller space $T_{g,n}$; such an algebra is an analog of the coordinate ring of $T_{g,n}$.

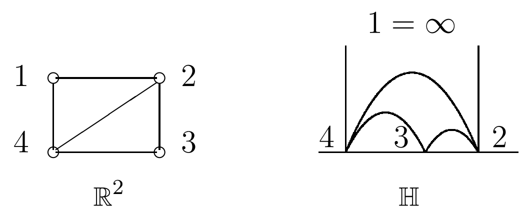

Figure 2: Triangulation of the Riemann surface $S_{1,1}$.

2.3 C^*-algebras

A *C^*-algebra* is an algebra A over $\mathbb{C}$ with a norm $a \mapsto ||a||$ and an involution $a \mapsto a^*$ such that it is complete with respect to the norm and $||ab|| \leq ||a||\,||b||$ and $||a^*a|| = ||a^2||$ for all $a, b \in A$. Any commutative C^*-algebra is isomorphic to the algebra $C_0(X)$ of continuous complex-valued functions on some locally compact Hausdorff space X; otherwise, A represents a noncommutative topological space.

An *AF-algebra* (Approximately Finite C^*-algebra) is defined to be the norm closure of an ascending sequence of finite dimensional C^*-algebras M_n, where M_n is the C^*-algebra of the $n \times n$ matrices with entries in $\mathbb{C}$. Here the index $n = (n_1, \ldots, n_k)$ represents the semi-simple matrix algebra $M_n = M_{n_1} \oplus \ldots \oplus M_{n_k}$. The ascending sequence mentioned above can be written as

$$M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} \ldots, \tag{7}$$

where M_i are the finite dimensional C^*-algebras and φ_i the homomorphisms between such algebras. The homomorphisms φ_i can be arranged into a graph as follows. Let $M_i = M_{i_1} \oplus \ldots \oplus M_{i_k}$ and $M_{i'} = M_{i'_1} \oplus \ldots \oplus M_{i'_k}$ be the semi-simple C^*-algebras and $\varphi_i : M_i \to M_{i'}$ the homomorphism. One has two sets of vertices $V_{i_1}, \ldots, V_{i_k}$ and $V_{i'_1}, \ldots, V_{i'_k}$ joined by b_{rs} edges whenever the summand M_{i_r} contains b_{rs} copies of the summand $M_{i'_s}$ under the embedding φ_i. As i varies, one obtains an infinite graph called the *Bratteli diagram* of the AF-algebra. The matrix $B = (b_{rs})$ is known as a *partial multiplicity* matrix; an infinite sequence of B_i defines a unique AF-algebra.

Let $\theta \in \mathbb{R}^{n-1}$; recall that by the *Jacobi-Perron continued fraction* of vector

$(1,\theta)$ one understands the limit:

$$\begin{pmatrix} 1 \\ \theta_1 \\ \vdots \\ \theta_{n-1} \end{pmatrix} = \lim_{k\to\infty} \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & b_1^{(1)} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & b_{n-1}^{(1)} \end{pmatrix} \dots \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & b_1^{(k)} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & b_{n-1}^{(k)} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

where $b_i^{(j)} \in \mathbb{N} \cup \{0\}$, see e.g. [Bernstein 1971] [1]; the limit converges for a generic subset of vectors $\theta \in \mathbb{R}^{n-1}$. Notice that $n = 2$ corresponds to (a matrix form of) the regular continued fraction of θ; such a fraction is always convergent. Moreover, the Jacobi-Perron fraction is *finite* if and only if vector $\theta = (\theta_i)$, where θ_i are *rational*. The AF-algebra A_θ *associated* to the vector $(1,\theta)$ is defined by the Bratteli diagram with the partial multiplicity matrices equal to B_k in the Jacobi-Perron fraction of $(1,\theta)$; in particular, if $n = 2$ the A_θ coincides with the Effros-Shen algebra [Effros & Shen 1980] [8].

2.4 Cluster C^*-algebras

Notice that the mutation tree $\overrightarrow{\mathbb{T}}_m$ of a cluster algebra $\mathcal{A}(\mathbf{x}, B)$ has a grading by levels, i.e. a distance from the root of $\overrightarrow{\mathbb{T}}_m$. We shall say that a pair of clusters $\mathbf{x}$ and $\mathbf{x}'$ are *ℓ-equivalent*, if:

(i) $\mathbf{x}$ and $\mathbf{x}'$ lie at the same level;

(ii) $\mathbf{x}$ and $\mathbf{x}'$ coincide modulo a cyclic permutation of variables x_i;

(iii) $B = B'$.

It is not hard to see that ℓ is an equivalence relation on the set of vertices of graph $\overrightarrow{\mathbb{T}}_m$.

Definition 1 *By a cluster C^*-algebra $\mathbb{A}(\mathbf{x}, B)$ one understands an AF-algebra given by the Bratteli diagram $\mathfrak{B}(\mathbf{x}, B)$ of the form:*

$$\mathfrak{B}(\mathbf{x}, B) := \overrightarrow{\mathbb{T}}_m \ \mathbf{mod}\ \ell. \tag{8}$$

The rank of $\mathbb{A}(\mathbf{x}, B)$ is equal to such of cluster algebra $\mathcal{A}(\mathbf{x}, B)$.

Example 3 If B_T is matrix (6) of Example 2, then $\mathfrak{B}(\mathbf{x}, B_T)$ is shown Figure 3. (We refer the reader to Section 4 for a proof.) Notice that the graph $\mathfrak{B}(\mathbf{x}, B_T)$ is a part of of the Bratteli diagram of the Mundici algebra $\mathfrak{M}$, compare [Mundici 2011, Figure 1] [13].

Remark 6 It is not hard to see that $\mathfrak{B}(\mathbf{x}, B)$ is no longer a tree and $\mathfrak{B}(\mathbf{x}, B)$ is a finite graph if and only if $\mathcal{A}(\mathbf{x}, B)$ is a finite cluster algebra.

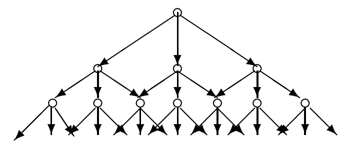

Figure 3: The Bratteli diagram of Markov's cluster C^*-algebra.

3 Proof

Let $m = 3(2g-2+n)$ be the rank of cluster C^*-algebra $\mathbb{A}(\mathbf{x}, Q_{g,n})$. For the sake of clarity, we shall consider the case $m=3$ and the general case $m \in \{3, 6, 9, \dots\}$ separately.

(i) Let $\mathbb{A}(\mathbf{x}, B_T)$ be the cluster C^*-algebra of rank 3. In this case $2g-2+n=1$ and either $g=0$ and $n=3$ or else $g=n=1$. Since $T_{0,3} \cong \{pt\}$ is trivial, we are left with $g=n=1$, i.e. the once-punctured torus $S_{1,1}$.

Repeating the argument of Example 2, we get the seed $(\mathbf{x}, B_T)$, where $\mathbf{x} = (x_1, x_2, x_3)$ and the skew-symmetric matrix B_T is given by formula (6).

Let us verify that matrix B_T is mutation finite; indeed, for each $k \in \{1,2,3\}$ the matrix mutation formula (4) gives us $\mu_k(B_T) = -B_T$.

Therefore, the exchange relations (3) do not vary; it is verified directly that such relations have the form:

$$\left\{\begin{array}{lll} x_1x_1' &=& x_2^2 + x_3^2, \\ x_2x_2' &=& x_1^2 + x_3^2, \\ x_3x_3' &=& x_1^2 + x_2^2. \end{array}\right. \tag{9}$$

Consider a mutation tree $\overrightarrow{\mathbb{T}}_3$ shown in Figure 4; the vertices of $\overrightarrow{\mathbb{T}}_3$ correspond to the mutations of cluster $\mathbf{x} = (x_1, x_2, x_3)$ following the exchange rules (9).

The reader is encouraged to verify that modulo a cyclic permutation of variables $x_1' = x_2, x_2' = x_3, x_3' = x_1$ and $x_1' = x_3, x_2' = x_1, x_3' = x_2$ one obtains (respectively) the following equivalences of clusters:

$$\left\{\begin{array}{lll} \mu_{13}(\mathbf{x}) &=& \mu_{21}(\mathbf{x}), \\ \mu_{23}(\mathbf{x}) &=& \mu_{31}(\mathbf{x}), \end{array}\right. \tag{10}$$

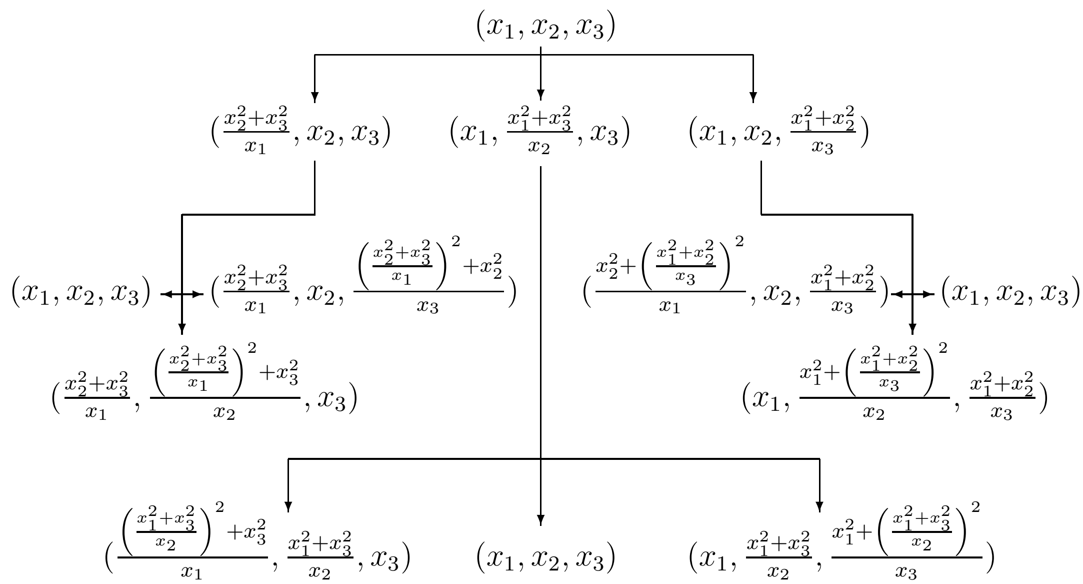

Figure 4: The mutation tree.

where $\mu_{ij}(\mathbf{x}) := \mu_j(\mu_i(\mathbf{x}))$; there are no other cluster equivalences for the vertices of the same level of graph $\overrightarrow{\mathbb{T}}_3$.

To determine the graph $\mathfrak{B}(\mathbf{x}, B_T)$ one needs to take the quotient of $\overrightarrow{\mathbb{T}}_3$ by the ℓ-equivalence relations (10); since the pattern repeats for each level of $\overrightarrow{\mathbb{T}}_3$, one gets the $\mathfrak{B}(\mathbf{x}, B_T)$ shown in Figure 3. The cluster C^*-algebra $\mathbb{A}(\mathbf{x}, B_T)$ is an AF-algebra with the Bratteli diagram $\mathfrak{B}(\mathbf{x}, B_T)$.

Notice that the Bratteli diagram $\mathfrak{B}(\mathbf{x}, B_T)$ of our AF-algebra $\mathbb{A}(\mathbf{x}, B_T)$ and such of the Mundici algebra $\mathfrak{M}$ are distinct, compare [Mundici 2011, Figure 1] [13]; yet there is an obvious inclusion of one diagram into another. Namely, if one erases a "camel's back" (i.e. the two extreme sides of the diagram) in the Bratteli diagram of $\mathfrak{M}$, then one gets exactly the diagram in Figure 3. Formally, if $\mathcal{G}$ is the Bratteli diagram of the Mundici algebra $\mathfrak{M}$, the complement $\mathcal{G} - \mathfrak{B}(\mathbf{x}, B_T)$ is a hereditary Bratteli diagram which gives rise to an ideal $I_0 \subset \mathfrak{M}$, such that:

$$\mathbb{A}(\mathbf{x}, B_T) \cong \mathfrak{M}/I_0, \tag{11}$$

see [Bratteli 1972, Lemma 3.2] [4]; the I_0 is a primitive ideal *ibid.*, Theorem

3.8. (It is interesting to calculate the group $K_0(I_0)$ in the context of the work of [Panti 1999] [18].)

On the other hand, the space $Prim\ \mathfrak{M}$ (and hence $Prim\ \mathbb{A}(\mathbf{x}, B_T)$) is well understood, see e.g. [Panti 1999] [18] or [Boca 2008, Proposition 7] [3]. Namely,

$$Prim\ (\mathfrak{M}/I_0) = \{I_\theta \mid \theta \in \mathbb{R}\}, \tag{12}$$

where $I_\theta \subset \mathfrak{M}$ is such that $\mathfrak{M}/I_\theta \cong A_\theta$ is the Effros-Shen algebra [Effros & Shen 1980] [8] if θ is an irrational number or $\mathfrak{M}/I_\theta \cong M_q$ is finite-dimensional matrix C^*-algebra (and an extension of such by the C^*-algebra of compact operators) if $\theta = \frac{p}{q}$ is a rational number. (Note that the third series of primitive ideals of [Boca 2008, Proposition 7] [3] correspond to the ideal I_0.) Moreover, given the Jacobson topology on $Prim\ \mathfrak{M}$, there exists a homeomorphism

$$h : Prim\ (\mathfrak{M}/I_0) \to \mathbb{R} \tag{13}$$

defined by the formula $I_\theta \mapsto \theta$, see [Boca 2008, Corollary 12] [3].

Let $\sigma_t : \mathfrak{M}/I_0 \to \mathfrak{M}/I_0$ be the Tomita-Takesaki flow, i.e. a one-parameter automorphism group of $\mathfrak{M}/I_0$, see Section 4. Because $I_\theta \subset \mathfrak{M}/I_0$, the image $\sigma^t(I_\theta)$ of I_θ is correctly defined for all $t \in \mathbb{R}$; the $\sigma_t(I_\theta)$ is an ideal of $\mathfrak{M}/I_0$ but not necessarily primitive. Since σ_t is nothing but (an algebraic form of) the Teichmüller geodesic flow on $T_{1,1}$ [Veech 1986] [21], one concludes that that the family of ideals

$$\{\sigma_t(I_\theta) \subset \mathfrak{M}/I_0 \mid t \in \mathbb{R},\ \theta \in \mathbb{R}\} \tag{14}$$

can be taken for a coordinate system in the space $T_{1,1} \cong \mathbb{R}^2$. In view of (13) and $\mathfrak{M}/I_0 \cong \mathbb{A}(\mathbf{x}, Q_{1,1})$, one gets the required homeomorphism

$$h : Prim\ \mathbb{A}(\mathbf{x}, Q_{1,1})\ \times\ \mathbb{R} \to T_{1,1}, \tag{15}$$

such that the quotient algebra $\mathbb{A}(\mathbf{x}, Q_{1,1})/\sigma_t(I_\theta)$ is a non-commutative coordinate ring of the Riemann surface $S_{1,1}$.

Remark 7 The family of algebras $\{\mathbb{A}(\mathbf{x}, Q_{1,1})/\sigma_t(I_\theta) \mid \theta = Const,\ t \in \mathbb{R}\}$ are in general pairwise non-isomorphic. (For otherwise all ideals $\{\sigma_t(I_\theta) \mid t \in \mathbb{R}\}$ were primitive.) Yet their Grothendieck semi-groups K_0^+ are, see [Effros & Shen 1980] [8]; the action of σ_t is given by the formula (see Section 4):

$$K_0^+(\mathbb{A}(\mathbf{x}, Q_{1,1})/\sigma_t(I_\theta)) \cong e^t(\mathbb{Z} + \mathbb{Z}\theta). \tag{16}$$

(ii) The general case $m = 3k = 3(2g - 2 + n)$ is treated likewise. Notice that if $d = 6g - 6 + 2n$ is dimension of the space $T_{g,n}$, then we have $m - d = n$; in particular, rank m of the cluster C^*-algebra $\mathbb{A}(\mathbf{x}, Q_{g,n})$ determines completely the pair (g, n) provided d is a fixed constant. (If d is not fixed, there is only a finite number of different pairs (g, n) for given rank m.)

Let $(\mathbf{x}, B_T)$ be the seed given by the cluster $\mathbf{x} = (x_1, \ldots, x_{3k})$ and the skew-symmetric matrix B_T. Since matrix B_T comes from a triangulation of the Riemann surface $S_{g,n}$, B_T is *mutation finite*, see [Williams 2014, p.18] [22]; the exchange relations (3) take the form:

$$\left\{\begin{array}{ccc} x_1 x_1' & = & x_2^2 + x_3^2 + \ldots + x_{3k}^2, \\ x_2 x_2' & = & x_1^2 + x_3^2 + \ldots + x_{3k}^2, \\ \vdots & & \\ x_{3k} x_{3k}' & = & x_1^2 + x_2^2 + \ldots + x_{3k-1}^2. \end{array}\right. \tag{17}$$

One can construct the mutation tree $\overrightarrow{\mathbb{T}}_{3k}$ using relations (17); the reader is encouraged to verify, that the $\overrightarrow{\mathbb{T}}_{3k}$ is similar to the one shown in Figure 4, except for the number of the outgoing edges at each vertex is equal to $3k$.

A tedious but straightforward calculation shows that the only equivalent clusters at the same level of $\overrightarrow{\mathbb{T}}_{3k}$ are the ones at the extremities of tuples $(x_1', \ldots, x_{3k}')$; in other words, one gets the following system of equivalences of clusters:

$$\left\{\begin{array}{ccc} \mu_{1,3k}(\mathbf{x}) & = & \mu_{21}(\mathbf{x}), \\ \mu_{2,3k}(\mathbf{x}) & = & \mu_{31}(\mathbf{x}), \\ \vdots & & \\ \mu_{3k-1,3k}(\mathbf{x}) & = & \mu_{3k,1}(\mathbf{x}), \end{array}\right. \tag{18}$$

where $\mu_{ij}(\mathbf{x}) := \mu_j(\mu_i(\mathbf{x}))$.

The graph $\mathfrak{B}(\mathbf{x}, B_T)$ is the quotient of $\overrightarrow{\mathbb{T}}_{3k}$ by the ℓ-equivalence relations (18); for $k = 2$ such a graph is sketched in Figure 5. The $\mathbb{A}(\mathbf{x}, Q_{g,n})$ is an AF-algebra given by the Bratteli diagram $\mathfrak{B}(\mathbf{x}, B_T)$.

Lemma 1 *The set*

$$Prim\ \mathbb{A}(\mathbf{x}, Q_{g,n}) = \{I_\theta \mid \theta \in \mathbb{R}^{6g-7+2n} \text{ is generic}\}, \tag{19}$$

where $\mathbb{A}(\mathbf{x}, Q_{g,n})/I_\theta$ is an AF-algebra A_θ associated to the convergent Jacobi-Perron continued fraction of vector $(1, \theta)$, see Section 2.3.

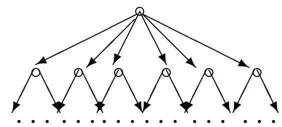

Figure 5: The Bratteli diagram of a cluster C^*-algebra of rank 6.

Proof. We adapt the argument of [Boca 2008, case $k = 1$] [3] to the case $k \geq 1$. Let $d = 6g - 6 + 2n$ be dimension of the space $T_{g,n}$. Roughly speaking, the Bratteli diagram $\mathfrak{B}(\mathbf{x}, B_T)$ of algebra $\mathbb{A}(\mathbf{x}, Q_{g,n})$ can be cut in two disjoint pieces $\mathcal{G}_\theta$ and $\mathfrak{B}(\mathbf{x}, B_T) - \mathcal{G}_\theta$, as it is shown by [Boca 2008, Figure 7] [3]. The $\mathcal{G}_\theta$ is a (finite or infinite) vertical strip of constant "width" d, where d is equal to the number of vertices cut from each level of $\mathfrak{B}(\mathbf{x}, B_T)$. The reader is encouraged to verify, that $\mathcal{G}_\theta$ is exactly the Bratteli diagram of the AF-algebra A_θ associated to the convergent Jacobi-Perron continued fraction of a *generic* vector $(1, \theta)$, see Section 2.3.

On the other hand, the complement $\mathfrak{B}(\mathbf{x}, B_T) - \mathcal{G}_\theta$ is a hereditary Bratteli diagram, which defines an ideal I_θ of algebra $\mathbb{A}(\mathbf{x}, Q_{g,n})$, such that:

$$\mathbb{A}(\mathbf{x}, Q_{g,n})/I_\theta = A_\theta, \tag{20}$$

see [Bratteli 1972, Lemma 3.2] [4]. Moreover, I_θ is a primitive ideal [Bratteli 1972, Theorem 3.8] [4]. (An extra care is required if $\theta = (\theta_i)$ is a rational vector; the complete argument can be found in [Boca 2008, pp. 980-985] [3].) Lemma 1 follows.

Lemma 2 *The sequence of primitive ideals I_{θ_n} converges to I_θ in the Jacobson topology in Prim $\mathbb{A}(\mathbf{x}, Q_{g,n})$ if and only if the sequence θ_n converges to θ in the Euclidean space $\mathbb{R}^{6g-7+2n}$.*

Proof. The proof is a straightforward adaption of the argument in [Boca 2008, pp. 986-988] [3]; we leave it as an exercise to the reader.

Let $\sigma_t : \mathbb{A}(\mathbf{x}, Q_{g,n}) \to \mathbb{A}(\mathbf{x}, Q_{g,n})$ be the Tomita-Takesaki flow, i.e. the group $\{\sigma_t \mid t \in \mathbb{R}\}$ of modular automorphisms of algebra $\mathbb{A}(\mathbf{x}, Q_{g,n})$, see Section 4. Because $I_\theta \subset \mathbb{A}(\mathbf{x}, Q_{g,n})$, the image $\sigma^t(I_\theta)$ of I_θ is correctly defined for all $t \in \mathbb{R}$; the $\sigma_t(I_\theta)$ is an ideal of $\mathbb{A}(\mathbf{x}, Q_{g,n})$ but not necessarily a primitive

ideal. Since σ_t is an algebraic form of the Teichmüller geodesic flow on the space $T_{g,n}$ [Veech 1986] [21], one concludes that that the family of ideals:

$$\{\sigma_t(I_\theta) \subset \mathbb{A}(\mathbf{x}, Q_{g,n}) \mid t \in \mathbb{R},\ \theta \in \mathbb{R}^{6g-7+2n}\} \tag{21}$$

can be taken for a coordinate system in the space $T_{g,n} \cong \mathbb{R}^{6g-6+2n}$. In view of Lemmas 1 and 2, one gets the required homeomorphism

$$h : Prim\ \mathbb{A}(\mathbf{x}, Q_{g,n}) \times \mathbb{R} \to \{U \subseteq T_{g,n} \mid U \text{ is generic}\}, \tag{22}$$

such that the quotient algebra $A_\theta = \mathbb{A}(\mathbf{x}, Q_{g,n})/\sigma_t(I_\theta)$ is a non-commutative coordinate ring of the Riemann surface $S_{g,n}$.

Theorem 1 is proved.

4 An analog of modular flow on $\mathbb{A}(\mathbf{x}, Q_{g,n})$

A. Modular automorphisms $\{\sigma_t \mid t \in \mathbb{R}\}$. Recall that the Ptolemy relations (5) for the Penner coordinates $\{\lambda(\gamma_i)\}$ in the space $T_{g,n}$ are homogeneous; in particular, the system $\{t\lambda(\gamma_i) \mid t \in \mathbb{R}\}$ of such coordinates will also satisfy the Ptolemy relations. On the other hand, for the cluster C^*-algebra $\mathbb{A}(\mathbf{x}, Q_{g,n})$ the variables $x_i = \lambda(\gamma_i)$ and one gets an obvious isomorphism $\mathbb{A}(\mathbf{x}, Q_{g,n}) \cong \mathbb{A}(t\mathbf{x}, Q_{g,n})$ for all $t \in \mathbb{R}$. Since $\mathbb{A}(t\mathbf{x}, Q_{g,n}) \subseteq \mathbb{A}(\mathbf{x}, Q_{g,n})$, one obtains a one-parmeter group of automorphisms:

$$\sigma_t : \mathbb{A}(\mathbf{x}, Q_{g,n}) \longrightarrow \mathbb{A}(\mathbf{x}, Q_{g,n}). \tag{23}$$

By analogy with [Connes 1978] [5], we shall call σ_t a *Tomita-Takesaki flow* on the cluster C^*-algebra $\mathbb{A}(\mathbf{x}, Q_{g,n})$. The reader is encouraged to verify, that σ_t is an algebraic form of the *geodesic flow* T^t on the Teichmüller space $T_{g,n}$, see [Veech 1986] [21] for an introduction. Roughly speaking, such a flow comes from the one-parameter group of matrices

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \tag{24}$$

acting on the space of holomorphic quadratic differentials on the Riemann surface $S_{g,n}$; the latter is known to be isomorphic to the Teichmüller space $T_{g,n}$.

B. Connes invariant $T(\mathbb{A}(\mathbf{x}, Q_{g,n}))$. Recall that an analogy of the *Connes invariant* $T(\mathcal{M})$ for a C^*-algebra $\mathcal{M}$ endowed with a modular automorphism group σ_t is the set $T(\mathcal{M}) := \{t \in \mathbb{R} \mid \sigma_t \text{ is inner}\}$ [Connes 1978] [5]. The group of inner automorphisms of the space $T_{g,n}$ and algebra $\mathbb{A}(\mathbf{x}, Q_{g,n})$ is isomorphic to the mapping class group $Mod\ S_{g,n}$ of surface $S_{g,n}$. The automorphism $\phi \in Mod\ S_{g,n}$ is called *pseudo-Anosov*, if $\phi(\mathcal{F}_\mu) = \lambda_\phi \mathcal{F}_\mu$, where $\mathcal{F}_\mu$ is an invariant measured foliation and $\lambda_\phi > 1$ is a constant called *dilatation* of ϕ; the λ_ϕ is always an algebraic number of the maximal degree $6g - 6 + 2n$ [Thurston 1988] [20]. It is known, that if $\phi \in Mod\ S_{g,n}$ is pseudo-Anosov then there exists a trajectory $\mathcal{O}$ of the geodesic flow T^t and a point $S_{g,n} \in T_{g,n}$, such that the points $S_{g,n}$ and $\phi(S_{g,n})$ belong to $\mathcal{O}$ [Veech 1986] [21]; the $\mathcal{O}$ is called an *axis* of the pseudo-Anosov automorphism ϕ. The axis can be used to calculate the Connes invariant $T(\mathbb{A}(\mathbf{x}, Q_{g,n}))$ of the cluster C^*-algebra $\mathbb{A}(\mathbf{x}, Q_{g,n})$; indeed, in view of formula (24) one must solve the following system of equations:

$$\left\{ \begin{array}{lcl} \sigma_t(x) & = & e^t x \\ \phi(x) & = & \lambda_\phi x, \end{array} \right. \tag{25}$$

for a point $x \in \mathcal{O}$. Thus $\sigma_t(x)$ coincides with the inner automorphism $\phi(x)$ if and only if $t = \log \lambda_\phi$. Taking all pseudo-Anosov automorphisms $\phi \in Mod\ S_{g,n}$, one gets a formula for the Connes invariant:

$$T(\mathbb{A}(\mathbf{x}, Q_{g,n})) = \{\log \lambda_\phi \mid \phi \in Mod\ S_{g,n} \text{ is pseudo-Anosov}\}. \tag{26}$$

Remark 8 The Connes invariant (26) says that the family of cluster C^*-algebras $\mathbb{A}(\mathbf{x}, Q_{g,n})$ is an analog of the type $\mathbf{III}_\lambda$ factors of von Neumann algebras, see [Connes 1978] [5].

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